The Burkholder-Davis-Gundy Inequality for Enhanced Martingales

Peter Friz Nicolas Victoir

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Abstract

Multi-dimensional continuous local martingales, enhanced with their stochastic area process, give rise to geometric rough paths with a.s. finite homogenous p-variation, $p > 2$. Here we go one step further and establish quantitative bounds of the p-variation norm in the form of a BDG inequality. Our proofs are based on old ideas by Lépingle. We also discuss geodesic and piecewise linear approximations.

1 Introduction

The theory of rough paths provides a new and robust way to drive differential equations by multi-dimensional stochastic processes in a *deterministic way*. In most cases, this is achieved by taking into account a certain stochastic area process and by establishing fine regularity properties of the resulting *enhanced* process. The object of study in this paper is a d-dimensional continuous local martingale M null at 0 for which the area is defined by iterated stochastic integration; the area process A_t is simply the anti-symmetric part of the iterated Stratonovich integral,

$$\mathbf{M}_t^2 \equiv \int_0^t \int_0^s dM_r \otimes \circ dM_s \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

Note that the symmetric part of $\mathbf{M}_t^2$ is given by $\frac{1}{2} M_t \otimes M_t$ and hence redundant if one knows $\mathbf{M}_t^1 \equiv M_t$. It follows that the enhanced process $\mathbf{M} \equiv \left(1, \mathbf{M}^1, \mathbf{M}^2\right) \in \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^d \otimes \mathbb{R}^d$ lives in submanifold, namely in $G^2\left(\mathbb{R}^d\right) \equiv \exp\left(\mathbb{R}^d \oplus so\left(d\right)\right)$, where $\exp : (x, a) \mapsto \left(1, x, a + \frac{1}{2} x \otimes x\right)$. The space $\mathbb{R}^d \oplus so\left(d\right)$ carries a Lie algebra structure and induces a (Lie-)group structure on $G^2\left(\mathbb{R}^d\right)$. The interest in this algebraic exercise is that the resulting product operation on $G^2\left(\mathbb{R}^d\right)$ is exactly what one needs to patch together "iterated integral increments" over adjacent intervals. $G^2\left(\mathbb{R}^d\right)$ is also a metric (in fact, Polish) space under the Carnot-Caratheodory metric d. Intuitively, the distance of two points under this metric is the length of the shortest path in $\mathbb{R}^d$ which wipes out a prescribed

area. When $d = 2$, geodesics are seen to be parts of circles. $G^2\left(\mathbb{R}^d\right)$ carries a dilation induced by $(x, a) \mapsto \left(\lambda x, \lambda^2 a\right)$ for real λ. In fact, the CC-metric is induced by a sub-additive norm, homogenous w.r.t. dilation. Since all continuous homogenous norms are Lipschitz equivalent, computations are often carried out w.r.t. $|||(x,a)||| = |x| + |a|^{1/2}$. We refer to [6, 7] for background on rough paths, [2] contains a more detailed discussion of the relevant geometry and algebra. The notion of a (weak) geometric p-rough path [3] becomes quite elegant: by definition, one requires that the $G^2\left(\mathbb{R}^d\right)$-valued path $\mathbf{M}$ has finite p-variation

$$\|\mathbf{M}\|_{p\text{-var};[0,T]} = \left(\sup_{0 \le t_1 \le \ldots \le t_n \le T} \sum d\left(\mathbf{M}_{t_i}, \mathbf{M}_{t_{i+1}}\right)^p\right)^{1/p} < \infty.$$

Is is known [1] that this holds for a.e. $\mathbf{M} = \mathbf{M}(\omega)$ when $p > 2$. The first main topic of this paper is to establish quantitative bounds of the p-variation norm in the form of a two-sided BDG inequality: for any moderate function F such as $x \mapsto x^r$ for $r > 0$,

$$\mathbb{E}\left(F\left(\|\mathbf{M}\|_{p\text{-var};[0,T}\right)\right) \sim \mathbb{E}\left(F\left(|\langle M \rangle_T|^{1/2}\right)\right).$$

The algebraic and geometric preparations made above prove crucial to recycle many of the arguments given in Lépingle's seminal paper [5] from 1976. Secondly, we discuss approximations and show L^q-convergence (at least for $q > 1$) of lifted piecewise linear approximations of a continuous L^q-martingale w.r.t. homogenous p-variation topology.

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2 Preliminaries

We write $\mathcal{M}^c_{0,\mathrm{loc}}\left([0,\infty), \mathbb{R}^d\right)$ or $\mathcal{M}^c_{0,\mathrm{loc}}\left(\mathbb{R}^d\right)$ for the class of $\mathbb{R}^d$-valued continuous local martingales $M : [0,\infty) \to \mathbb{R}^d$ null at 0. The bracket process $\langle M \rangle : [0,\infty) \to \mathbb{R}^d$ is defined component-wise, the i^{th} component is given by the usual bracket $\left\langle M^i \right\rangle = \left\langle M^i, M^i \right\rangle$.

The area-process $A : [0,\infty) \to so(d)$ is defined by Itô- or Stratonovich stochastic integration. As the matrix $\left\langle M^i, M^j \right\rangle$ is symmetric both lead to the *same* area,

$$\begin{aligned} A^{i,j}_t &= \frac{1}{2}\left(\int_0^t M^i dM^j - \int_0^t M^j dM^i\right) \\ &= \frac{1}{2}\left(\int_0^t M^i \circ dM^j - \int_0^t M^j \circ dM^i\right). \end{aligned}$$

We note that the area-process is a vector-valued continuous martingale. By disregarding a null-set we can and will assume that M and A are continuous.

Definition 1 *Set* $S_2(M) := \mathbf{M} := \exp(M + A)$ *so that* $\mathbf{M} \in C([0,\infty), G^2(\mathbb{R}^d))$. *The resulting class of enhanced (continuous, local) martingales is denoted by* $\mathcal{M}^c_{0,loc}(G^2(\mathbb{R}^d))$. *We refer to the operation* $S_2 : M \mapsto \mathbf{M}$ *as lift.*

The lift is compatible with the stopping and time-changes.

Lemma 2 *(i) Let* τ *be a stopping time. Then* $\mathbf{M}^\tau = S_2(M^\tau)$. *(ii) Let* ϕ *be a time-change, that is, a family* ϕ_s, $s \geq 0$, *of stopping times such that the maps* $s \mapsto \phi_s$ *are a.s. increasing and right-continuous. Assume that* M *is constant on each interval* $[\phi_{t-}, \phi_t]$. *Then* $M \circ \phi$ *is a continuous local martingale and*

$$\mathbf{M} \circ \phi = S_2(M \circ \phi).$$

Proof. Stopped processes are special cases of time-changed processes (take $\phi_t = t \wedge \tau$) so it suffices to show the second statement. To this end, recall the compatibility of a time change ϕ and stochastic integration w.r.t. a continuous local martingale, constant on each interval $[\phi_{t-}, \phi_t]$, Proposition V.1.5. (ii) of [8]. The lift is a special case of stochastic integration. ■

The lift is also compatible with respect to scaling and concatenation of (local martingale) paths.

Lemma 3 *(i) If* $\delta_c : G^2(\mathbb{R}^d) \to G^2(\mathbb{R}^d)$ *is the dilation operator given by* $\delta_c \exp(x + a) = \exp(cx + c^2 a)$ *then* $\delta_c \mathbf{M} = S_2(cM)$. *(ii) We have*

$$S_2(M)_{0,t} = S_2\left(M|_{[0,s]} * M|_{[s,t]} \right)_{0,t} = S_2(M)_{0,s} \otimes S_2(M)_{s,t}\,, \ \ 0 \leq s \leq t < \infty$$

Proof. (i) follows is trivial consequence of linearity of stochastic integrals. (ii) is true whenever a first order calculus underlies the lift. It now suffices to note that S_2 is (equivalently) defined as Stratonovich lift,

$$S_2(M)_t = \exp(M_t + A_t) = 1 + M_t + \int_0^t M_s \otimes \circ dM_s.$$

■

Definition 4 $F : \mathbb{R}^+ \to \mathbb{R}^+$ *is moderate if (i)* F *is continuous and increasing, (ii)* $F(x) = 0$ *if and only if* $x = 0$ *and (iii) for some (and then for every)* $\alpha > 1$,

$$\sup_{x>0} \frac{F(\alpha x)}{F(x)} < \infty.$$

The following result is found, for instance, in [9, p93].

Theorem 5 (Burkholder-Davis-Gundy) *Let* F *be a moderate function,* $M \in \mathcal{M}^c_{0,loc}(\mathbb{R})$. *Then there exists a constant* $C = C(F, d, |\cdot|)$ *so that*

$$C^{-1}\mathbb{E}\left(F\left(|\langle M\rangle_\infty|^{1/2}\right)\right) \leq \mathbb{E}\left(F\left(\sup_{s\geq 0}|M_s|\right)\right) \leq C\mathbb{E}\left(F\left(|\langle M\rangle_\infty|^{1/2}\right)\right).$$

We collect a few properties of moderate functions.

Lemma 6 *(i)* $x \mapsto F(x)$ *moderate iff* $\mapsto F\left(x^{1/2}\right)$ *moderate.*
(ii) Given $c, A, B > 0 : c^{-1}A \leq B \leq cA \implies \exists C = C(c, F)$:

$$C^{-1}F(A) \leq F(B) \leq CF(A).$$

(iii) $\exists C : \forall x, y > 0 : F(x+y) \leq C[F(x) + F(y)]$.

Proof. (i),(ii) are left to the reader. Ad (iii): W.l.o.g. $x < y$, then $F(x+y) \leq F(2y) \leq CF(y)$ by moderate growth of F. ■

Corollary 7 *Let* F *be a moderate function,* $M \in \mathcal{M}^c_{0,loc}\left(\mathbb{R}^d\right)$ *and* $|\cdot|$ *a norm on* $\mathbb{R}^d$. *Then there exists a constant* $C = C(F, d, |\cdot|)$ *so that*

$$C^{-1}\mathbb{E}\left(F\left(\left|\langle M\rangle_\infty\right|^{1/2}\right)\right) \leq \mathbb{E}\left(F\left(\sup_{s\geq 0}|M_s|\right)\right) \leq C\mathbb{E}\left(F\left(\left|\langle M\rangle_\infty\right|^{1/2}\right)\right).$$

Proof. When $|a| = \max\left\{\left|a^1\right|, ..., \left|a^d\right|\right\}$ this is a simple consequence of BDG for $\mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{R})$, applied componentwise. The lemma above shows that one can switch to Lipschitz equivalent norms. ■

From Lépingle [5], $\sup_{s\geq 0}|M_s|$ above can be replaced by the p-variation norm[1]. Noting that the p-variation of a discrete-time martingale (Y_n) is naturally defined as

$$|Y|_{p\text{-var}} \equiv \left[\sup_{(n_k)\nearrow}\sum_k \left|Y_{n_{k+1}} - Y_{n_k}\right|^p\right]^{1/p},$$

the following lemma is best viewed as a BDG-type upper bound for discrete-time martingales.

Lemma 8 *Let* F *be moderate. If* $1 < q < p \leq 2$ *or* $1 = q = p$ *then there exists a constant c such that for all, possibly* $\mathbb{R}^d$*-valued, discrete-time martingales* $(Y_n : n \in \mathbb{Z}^+)$

$$\mathbb{E}\left(F\left(|Y|_{p\text{-}var}\right)\right) \leq c\mathbb{E}\left[F\left(\left[\sum_n |Y_{n+1} - Y_n|^q\right]^{1/q}\right)\right].$$

Proof. For $d = 1$ we can use Proposition 2.b in [5] with the remark that a discrete-time martingale can be viewed as a particular case of a continuous-time martingale with purely discontinuous sample paths. As above, the extension to $d > 1$ does not pose any difficulty. ■

[1] In the next section, we will see a more general version of this.

3 BDG on the group

Lemma 9 (Good λ inequality, [9, p.94]) *Let X, Y be nonnegative random variables, and suppose there exists $\beta > 1$ such that for all $\lambda > 0, \delta > 0$,*

$$\mathbb{P}(X > \beta\lambda, Y < \delta\lambda) \leq \psi(\delta)\,\mathbb{P}(X > \lambda)$$

where $\psi(\delta) \searrow 0$ when $\delta \searrow 0$. There, for each moderate function F, there exists a constant C depending only on β, ψ, F such that

$$\mathbb{E}(F(X)) \leq C\mathbb{E}(F(Y)).$$

Proposition 10 *Let $|\cdot|, \|\cdot\|$ continuous homogonous norm on $\mathbb{R}^d, G^2(\mathbb{R}^d)$ respectively. Then there exists a constant $A = A(d, |\cdot|, \|\cdot\|)$ such that*

$$\forall \mathbf{M} \in \mathcal{M}^c_{0,loc}\left(G^2\left(\mathbb{R}^d\right)\right) \forall\lambda > 0 : \mathbb{P}\left(\sup_{u,v\geq 0} \|\mathbf{M}_{u,v}\| \geq \lambda\right) \leq A\frac{\mathbb{E}\left(|\langle M\rangle_\infty|\right)}{\lambda^2}.$$

Proof. We note that $\sup_{u,v\geq 0}\|\mathbf{M}_{u,v}\| \leq 2\sup_{t\geq 0}\|\mathbf{M}_t\|$. By equivalence of homogeneous norm,

$$\|\mathbf{M}_t\|^2 \leq C\left(|M_t|^2 + |A_t|\right).$$

From BDG, $\mathbb{E}\left(\sup_{u\geq 0}|M_u|^2\right) \leq C\mathbb{E}(|\langle M\rangle_\infty|)$. Note that $u \mapsto |\langle M\rangle|_u := \sum_{i=1}^d \langle M^i\rangle_u$ is increasing (in fact, there is no loss in generality in assuming that $|\cdot|$ on $\mathbb{R}^d$ is given given by $|a| = \sum |a^i|$ ...). Then, using BDG again,

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\geq 0}|A_u|\right) &\leq C\mathbb{E}\left(\left|\int_0^\infty |M_u|^2\, d\,|\langle M\rangle|_u\right|^{1/2}\right) \\
&\leq C\mathbb{E}\left(\sup_{u\geq 0}|M_u| \,.\, |\langle M\rangle|_\infty^{1/2}\right) \\
&\leq C\sqrt{\mathbb{E}\sup_{u\geq 0}|M_u|^2}\sqrt{\mathbb{E}\left[|\langle M\rangle|_\infty\right]} \\
&\leq C\mathbb{E}\left(|\langle M\rangle|_\infty\right).
\end{aligned}$$

An application of Chebyshev's inequality finishes the proof. ■

Theorem 11 *Let F be a moderate function, $\mathbf{M} \in \mathcal{M}^c_{0,loc}\left(G^2\left(\mathbb{R}^d\right)\right)$, and $|\cdot|, \|\cdot\|$ continuous homogonous norm on $\mathbb{R}^d, G^2(\mathbb{R}^d)$ respectively. Then there exists a constant $C = C(F, d, |\cdot|, \|\cdot\|)$ so that*

$$C^{-1}\mathbb{E}\left(F\left(|\langle M\rangle_\infty|^{1/2}\right)\right) \leq \mathbb{E}\left(F\left(\sup_{s,t\geq 0}\|\mathbf{M}_{s,t}\|\right)\right) \leq C\mathbb{E}\left(F\left(|\langle M\rangle_\infty|^{1/2}\right)\right).$$

Proof. The lower bound comes from

$$\|\mathbf{M}_{s,t}\| \geq |M_{s,t}|$$

the monotonicity of F and the classical BDG lower bound. We prove the upper-bound: we fix $\lambda, \delta > 0$ and $\beta > 1$, and we define the stopping times

$$\begin{aligned} S_1 &= \inf\left\{ t > 0, \sup_{u,v \in [0,t]} \|\mathbf{M}_{u,v}\| > \beta\lambda \right\}, \\ S_2 &= \inf\left\{ t > 0, \sup_{u,v \in [0,t]} \|\mathbf{M}_{u,v}\| > \lambda \right\}, \\ S_3 &= \inf\left\{ t > 0, \left|\langle M \rangle_t\right|^{1/2} > \delta\lambda \right\}, \end{aligned}$$

with the convention that the infimum of the empty set if ∞. Define the local martingale $N_t = M_{S_3 \wedge S_2, (t+S_2) \wedge S_3}$noting that $N_t \equiv 0$ on $\{S_2 = \infty\}$. It is easy to see that

$$\sup_{u,v \in [0,S_3]} \|\mathbf{M}_{u,v}\| \leq \sup_{u,v \in [0,S_3 \wedge S_2]} \|\mathbf{M}_{u,v}\| + \sup_{u,v \geq 0} \|\mathbf{N}_{u,v}\|. \tag{1}$$

By definition of the relevant stopping times,

$$\mathbb{P}\left(\sup_{u,v \geq 0} \|\mathbf{M}_{u,v}\| > \beta\lambda, \left|\langle M \rangle_\infty\right|^{1/2} \leq \delta\lambda \right) = \mathbb{P}\left(S_1 < \infty, S_3 = \infty\right).$$

On the event $\{S_1 < \infty, S_3 = \infty\}$ one has

$$\sup_{u,v \in [0,S_3]} \|\mathbf{M}_{u,v}\| > \beta\lambda$$

and, since $S_2 \leq S_1$, one also has $\sup_{u,v \in [0,S_3 \wedge S_2]} \|\mathbf{M}_{u,v}\| = \lambda$. Hence, on $\{S_1 < \infty, S_3 = \infty\}$,

$$\sup_{u,v \geq 0} \|\mathbf{N}_{u,v}\| \geq \sup_{u,v \in [0,S_3]} \|\mathbf{M}_{u,v}\| - \sup_{u,v \in [0,S_3 \wedge S_2]} \|\mathbf{M}_{u,v}\| \geq (\beta - 1)\lambda.$$

Therefore, using Proposition 10,

$$\begin{aligned} \mathbb{P}\left(\sup_{u,v \geq 0} \|\mathbf{M}_{u,v}\| > \beta\lambda, \left|\langle M \rangle_\infty\right|^{1/2} \leq \delta\lambda \right) &\leq \mathbb{P}\left(\sup_{u,v \geq 0} \|\mathbf{N}_{u,v}\| \geq (\beta - 1)\lambda \right) \\ &\leq \frac{A}{(\beta - 1)^2 \lambda^2} \mathbb{E}\left(\left|\langle N \rangle_\infty\right|\right). \end{aligned}$$

From the definition of N, for every $t \in [0, \infty]$,

$$\langle N \rangle_t = \langle M \rangle_{S_3 \wedge S_2, (t+S_2) \wedge S_3}.$$

On $\{S_2 = \infty\}$ we have $\langle N\rangle_\infty = 0$ while on $\{S_2 < \infty\}$ we have, from definition of S_3,

$$|\langle N\rangle_\infty| = \left|\langle M\rangle_{S_3\wedge S_2, S_3}\right| = \left|\langle M\rangle_{S_3} - \langle M\rangle_{S_3\wedge S_2}\right| \leq 2\left|\langle M\rangle_{S_3}\right| = 2\delta^2\lambda^2.$$

It follows that

$$\mathbb{E}\left(|\langle N\rangle_\infty|\right) \leq 2\delta^2\lambda^2\mathbb{P}\left(S_2 < \infty\right) = 2\delta^2\lambda^2\mathbb{P}\left(\sup_{u,v\geq 0}\|\mathbf{M}_{u,v}\| > \lambda\right)$$

and we have the estimate

$$\mathbb{P}\left(\sup_{u,v\geq 0}\|\mathbf{M}_{u,v}\| > \beta\lambda, |\langle M\rangle_\infty|^{1/2} \leq \delta\lambda\right) \leq \frac{2A\delta^2}{(\beta-1)^2}\mathbb{P}\left(\sup_{u,v\geq 0}\|\mathbf{M}_{u,v}\| > \lambda\right).$$

An application of the good λ-inequality finishes the proof. ■

4 Path regularity and p-variation BDG

Let $p > 2$. From [1] it is known that for every $\mathbf{M} \in \mathcal{M}^c_{0,\mathrm{loc}}\left(G^2\left(\mathbb{R}^d\right)\right)$ and every $T > 0$

$$\|\mathbf{M}\|_{p\text{-var};[0,T]} < \infty \text{ a.s.} \tag{2}$$

Here, we go one step further and provided quantitative bounds for the p-variation of the enhanced martingale in terms of $\langle M\rangle_T$. En passant, we give a simplified proof of (2).

Proposition 12 *Let $\mathbf{M} \in \mathcal{M}^c_{0,loc}\left(G^2\left(\mathbb{R}^d\right)\right)$. Then, for every $T > 0$,*

$$\|\mathbf{M}\|_{p\text{-}var;[0,T]} < \infty \ \textit{a.s.}$$

Proof. There exists a sequence of stopping times $\tau_n \to \infty$ a.s. such that M^{τ_n} and $\langle M^{\tau_n}\rangle$ are bounded (for instance, $\tau_n = \inf\{t : |M_t| > n \text{ or } |\langle M\rangle_t| > n\}$ will do.) Since

$$\mathbb{P}\left(\|\mathbf{M}\|_{p\text{-var};[0,T]} \neq \|\mathbf{M}\|_{p\text{-var};[0,T\wedge\tau_n]}\right) \leq \mathbb{P}\left(\tau_n < T\right) \to 0 \text{ as } n \to \infty$$

it suffices to consider the lift of a bounded continuous martingale with bounded quadratic variation. We can work with the l^1-norm on $\mathbb{R}^d$, $|a| = \sum_{i=1}^d |a_i|$. The time change $\phi(t) := \inf\{s : |\langle M\rangle_s| > t\}$ may have jumps but continuity of $|\langle M\rangle|$ ensures that $|\langle M\rangle_{\phi(t)}| = t$. From definition of ϕ and the BDG inequality on the group, both $|\langle M\rangle|$ and $\mathbf{M}$ are constant on the intervals $\left[\phi_{t-}, \phi_t\right]$. It follows that $\mathbf{X}_t = \mathbf{M}_{\phi(t)}$ defines a continuous[2] path from $[0, |\langle M\rangle_T|]$ to $G^2\left(\mathbb{R}^d\right)$ and it is easy to see that

$$\|\mathbf{X}\|_{p\text{-var},\left[0,|\langle M\rangle_T|\right]} = \|\mathbf{M}\|_{p\text{-var},[0,T]}.$$

[2]From Lemma 2, $\mathbf{X} = S_2(M\circ\phi)$, the lift of a continuous local martingale. In particular, this is another way to see continuity of $\mathbf{X}$.

As argued in the beginning of the proof, we may assume that $|\langle M\rangle_T| \le R$ for some deterministic R large enough. Therefore,

$$\begin{aligned}
\mathbb{P}\left(\|\mathbf{M}\|_{p\text{-var},[0,T]} > K\right) &= \mathbb{P}\left(\|\mathbf{X}\|_{p\text{-var},\left[0,\left|\langle M\rangle_T\right|\right]}, |\langle M\rangle_T| \le R\right) \qquad (3)\\
&\le \mathbb{P}\left(\|\mathbf{X}\|_{p\text{-var},[0,R]} > K\right).
\end{aligned}$$

We go on to show that $\mathbf{X}$ is in fact Hölder continuous. For $0 \le s \le t \le R$, we can use the BDG inequality on the group, theorem 11, to obtain

$$\mathbb{E}\left(\|\mathbf{X}_{s,t}\|^{2q}\right) = \mathbb{E}\left(\left\|\mathbf{M}_{\phi(s),\phi(t)}\right\|^{2q}\right) \le C_q \mathbb{E}\left(\left|\langle M\rangle_{\phi(t)} - \langle M\rangle_{\phi(s)}\right|^q\right).$$

Observe that

$$\begin{aligned}
\left|\langle M\rangle_{\phi(t)} - \langle M\rangle_{\phi(s)}\right| &= \sum_i \left(\langle M^i\rangle_{\phi(t)} - \langle M^i\rangle_{\phi(s)}\right)\\
&= \left|\langle M\rangle_{\phi(t)}\right| - \left|\langle M\rangle_{\phi(s)}\right| = t - s.
\end{aligned}$$

Thus, for all $q < \infty$ there exists a constant C_q s.t.

$$\mathbb{E}\left(\|\mathbf{X}_{s,t}\|^{2q}\right) \le C_q |t-s|^q.$$

Knowing that $\mathbf{X}$ is continuous, we can apply GRR[3] for paths in $\left(G^2\left(\mathbb{R}^d\right), d\right)$ to see that $\|\mathbf{X}\|_{1/p\text{-Hölder},[0,R]} \in L^q$ for all $q \in [1,\infty)$ and

$$\mathbb{P}\left(\|\mathbf{X}\|_{p\text{-var},[0,R]} > K\right) \le \frac{\mathbb{E}\left(\|\mathbf{X}\|_{p\text{-var},[0,R]}\right)}{K} \le \frac{\mathbb{E}\left(\|\mathbf{X}\|_{1/p\text{-Hölder},[0,R]}\right)}{K}$$

tends to zero as $K \to \infty$. Together with (3) we see that $\|\mathbf{M}\|_{p\text{-var},[0,T]} < \infty$ with probability 1 as claimed. ■

We are now going to prove a p-variation version of BDG. For $\mathbb{R}$-valued martingales this result is due to Lépingle, [5]. With the preparations made, our proof follows the same lines.

Lemma 13 *There exists a constant A such that for all continous local martingales M, for all $\lambda > 0$,*

$$\mathbb{P}\left(\|\mathbf{M}\|_{p\text{-}var;[0,\infty)} > \lambda\right) \le A \frac{\mathbb{E}\left(|\langle M\rangle_\infty|\right)}{\lambda^2}.$$

Proof. If suffices to prove the statement when $\lambda = 1$ (the general case follows by considering M/λ with lift $\delta_{1/\lambda}\mathbf{M}$). The statement then reduces to

$$\exists A : \forall M : \mathbb{P}\left[\|\mathbf{M}\|_{p\text{-var};[0,\infty)} > 1\right] \le A\,\mathbb{E}\left(|\langle M\rangle_\infty|\right).$$

[3] There is no modification of $\mathbf{X}$ needed.

Assume this is false. Then for every A, and in particular for $A(k) \equiv k^2$,there exists $M \equiv M^{(k)}$ with lift $\mathbf{M}^{(k)}$ s.t. the condition is violated,

$$k^2 \, \mathbb{E}\left[\left|\left\langle M^{(k)}\right\rangle_\infty\right|\right] < \mathbb{P}\left[\left\|\mathbf{M}^{(k)}\right\|_{p\text{-var};[0,\infty)} > 1\right] \leq 1.$$

Set $u_k = \mathbb{P}\left[\left\|\mathbf{M}^{(k)}\right\|_{p\text{-var};[0,\infty)} > 1\right]$, $n_k = [1/u_k + 1] \in \mathbb{N}$ and note that $1 \leq n_k u_k \leq 2$. Take n_k copies of each $M^{(k)}$ and get a sequence of martingales of form

$$\left(\tilde{M}\right) \equiv (\underbrace{M^{(1)}, ..., M^{(1)}}_{n_1}; \underbrace{M^{(2)}, ..., M^{(2)}}_{n_2}; M^{(3)}, ...).$$

Then

$$n_k k^2 \mathbb{E}\left[\left|\left\langle M^{(k)}\right\rangle_\infty\right|\right] \leq n_k \mathbb{P}\left[\left\|\mathbf{M}^{(k)}\right\|_{p\text{-var};[0,\infty)} > 1\right] = n_k u_k \leq 2.$$

and

$$\sum_k \mathbb{P}\left[\left\|\tilde{\mathbf{M}}^{(k)}\right\|_{p\text{-var};[0,\infty)} > 1\right] = \sum_k n_k u_k = +\infty$$

while

$$\sum_k \mathbb{E}\left[\left|\left\langle \tilde{M}^{(k)}\right\rangle_\infty\right|\right] = \sum_k n_k \mathbb{E}\left[\left|\left\langle M^{(k)}\right\rangle_\infty\right|\right] \leq \sum_k \frac{2}{k^2} < \infty.$$

Thus, if the claimed statement is false, there exists a sequence of martingales, we now revert to write $M^{(k)}, \mathbf{M}^{(k)}$ instead of $\tilde{M}^{(k)}, \tilde{\mathbf{M}}^{(k)}$ respectively, each defined on some filtered probability space $\left(\Omega^k, \left(\mathcal{F}_t^k\right), \mathbb{P}^k\right)$ with the two properties

$$\sum_k \mathbb{P}^k\left[\left\|\mathbf{M}^{(k)}\right\|_{p\text{-var};[0,\infty)} > 1\right] = +\infty \text{ and } \sum_k \mathbb{E}^k\left[\left|\left\langle M^{(k)}\right\rangle_\infty\right|\right] < \infty.$$

Define the probability space $\Omega = \bigotimes_{k=1}^\infty \Omega^k$, the probability $\mathbb{P} = \bigotimes_{k=1}^\infty \mathbb{P}^k$, and the filtration $(\mathcal{F}_t)$ on Ω given by

$$\mathcal{F}_t = \left(\bigotimes_{i=1}^{k-1} \mathcal{F}_\infty^i\right) \otimes \mathcal{F}_{g(k-t)}^k \otimes \left(\bigotimes_{j=k+1}^{\infty} \mathcal{F}_0^k\right) \quad \text{for } k-1 \leq t < k.$$

where $g(u) = 1/u - 1$ maps $[0,1] \to [0,\infty]$. Then, a continous martingale on $(\Omega, (\mathcal{F}_t), \mathbb{P})$ is defined by concatenation,

$$M_t = \sum_{i=1}^{k-1} M_\infty^{(i)} + M_{g(k-t)}^{(k)} \text{ for } \quad k-1 \leq t < k.$$

which implies

$$\mathbf{M}_t = \left(\bigotimes_{i=1}^{k-1} \mathbf{M}_\infty^{(i)}\right) \otimes \mathbf{M}_{g(k-t)}^{(k)}.$$

We also observe that, again for $k-1 \le t < k$,

$$\langle M \rangle_t = \sum_{i=1}^{k-1} \left\langle M^{(i)} \right\rangle_\infty + \left\langle M^{(k)} \right\rangle_{g(k-t)}.$$

In particular, $\langle M \rangle_\infty = \sum_k \left\langle M^{(k)} \right\rangle_\infty$ and, using the second property of the martingale sequence, $\mathbb{E}\left(\left|\langle M \rangle_\infty\right|\right) < \infty$. Define the events

$$A_k = \left\{ \|\mathbf{M}\|_{p\text{-var};[k-1,k]} > 1 \right\}.$$

Then, using the first property of the martingale sequence,

$$\sum_k \mathbb{P}(A_k) = \sum_k \mathbb{P}^k \left(\left\|\mathbf{M}^k\right\|_{p\text{-var};[0,\infty)} > 1 \right) = \infty.$$

Since the events $\{A_k : k \ge 1\}$ are independent, the Borel-Cantelli lemma implies that $\mathbb{P}(A_k \text{ i.o.}) = 1$. Thus, almost surely, for all $K > 0$ there exists a finite number of increasing times $t_0, \cdots, t_n \in [0,\infty)$ so that

$$\sum_{i=1}^{n} \left\| \mathbf{M}_{t_{i-1},t_i} \right\| > K$$

and $\|\mathbf{M}\|_{p\text{-var};[0,\infty)}$ must be equal to $+\infty$ with probability one. We now define a martingale N by time-change, namely via $f(t) = t/(1-t)$ for $0 \le t < 1$ and $f(t) = \infty$ for $t \ge 1$,

$$N : t \mapsto M_{f(t)}.$$

Note that $\mathbb{E}\left(\left|\langle M \rangle_\infty\right|\right) < \infty$ so that M can be extended to a (continuous) martingale indexed by $[0,\infty]$ and N is indeed a continuous martingale with lift $\mathbf{N}$. Since lifts interchange with time changes, $\|\mathbf{N}\|_{p\text{-var};[0,1]} = \|\mathbf{M}\|_{p\text{-var};[0,\infty)} = +\infty$ with probability one. But this contradicts to p-variation regularity result above. ■

The very same argument that was used in the proof of Theorem 11 now leads to the following BDG inequality for enhanced continuous local martingales w.r.t. homogenuous p-variation norm.

Theorem 14 *Let F be a moderate function, $\mathbf{M} \in \mathcal{M}^c_{0,loc}\left(G^2\left(\mathbb{R}^d\right)\right)$, and $|\cdot|$, $\|\cdot\|$ continuous homogonous norm on $\mathbb{R}^d, G^2\left(\mathbb{R}^d\right)$ respectively and $p > 2$. Then there exists a constant $C = C(p, F, d, |\cdot|, \|\cdot\|)$ so that*

$$C^{-1}\mathbb{E}\left(F\left(\left|\langle M \rangle_\infty\right|^{1/2}\right)\right) \le \mathbb{E}\left(F\left(\|\mathbf{M}\|_{p\text{-}var;[0,\infty)}\right)\right) \le C\mathbb{E}\left(F\left(\left|\langle M \rangle_\infty\right|^{1/2}\right)\right).$$

Remark 15 *When $p \in (2,3)$ and $N \in \{3,4,...\}$, $\mathbf{M}$ lifts uniquely to a $G^N\left(\mathbb{R}^d\right)$-valued path with finite homogenuous p-variation regularity, denoted by $S_N(\mathbf{M})$, which is identified with the first N iterated Stratonovich integrals of M. A basic theorem of Lyons asserts that*

$$\|S_N(\mathbf{M})\|_{p\text{-}var} \le C(N)\|\mathbf{M}\|_{p\text{-}var}$$

and BDG inequalities for the p-variation of this step-N lift are an immediate corollary of Theorem 14.

5 Approximations

We now only consider (lifted) local martingales on $[0,T]$, defined or identified with local martingales stopped at $T>0$.

5.1 Geodesic approximations

The p-variation norm of geodesics approximations is uniformly controlled by the original p-variation norm. Therefore

$$\mathbb{E}\left(F\left(\sup_D \left\|\mathbf{M}^{[D]}\right\|_{p\text{-var};[0,T]}\right)\right) \leq C\mathbb{E}\left(F\left(\left|\langle M\rangle_T\right|^{1/2}\right)\right)$$

where $\mathbf{M}^{[D]}$ denotes the geodesics approxiation to $\mathbf{M}$ based on some dissection D of $[0,T]$. Note that this is stronger than

$$\sup_D \mathbb{E}\left(F\left(\left\|\mathbf{M}^{[D]}\right\|_{p\text{-var};[0,T]}\right)\right) \leq C\mathbb{E}\left(F\left(\left|\langle M\rangle_T\right|^{1/2}\right)\right)$$

which is what we are going to show for piecewise linear approximations.

5.2 Piecewise linear approximations

Let $D=(t_i)$ be a subdivision of $[0,T]$. Given $x\in C\left([0,T],\mathbb{R}^d\right)$ we define x^D to be the piecewise linear approximation of x which coincides with x on D. Since x^D is of bounded variation, it admits a canonical lift to a $G^2\left(\mathbb{R}^d\right)$-valued path, denoted by $\mathbf{x}^D$. This notation applies path-by-path to $M\in\mathcal{M}^c_{0,\mathrm{loc}}\left(\mathbb{R}^d\right)$, we write $\mathbf{M}^D=\mathbf{M}^D(\omega)$ for the lifted piecewise linear approximation to $M(\omega)$. The next lemma involves no probabilty.

Lemma 16 *Set $\mathbf{x}^D=S_2\left(x^D\right)$ where x^D is linear between the points of D. Then there exists a constant $C=C=C(d,|\cdot|,\|\cdot\|)$ such that*

$$\left\|\mathbf{x}^D\right\|_{p-var;[0,T]} \leq C\left(\max_{(s_k)\subset D}\sum_k \left\|\mathbf{x}^D_{s_k,s_{k+1}}\right\|^p\right)^{1/p} + C\left|x\right|_{p\text{-}var;[0,T]}.$$

Proof. First we note that $\left\|\mathbf{x}^D_{s,t}\right\|^p \leq 3^{p-1}\left[\left|x^D_{s,s^D}\right|^p + \left\|\mathbf{x}^D_{s^D,t_D}\right\|^p + \left|x^D_{t_D,t}\right|^p\right]$. Now let (u_k) be a dissection of $[0,T]$, unrelated to D. Recall that u^D resp. u_D refers to the right- resp. left-neighbours of u in D.

$$\begin{aligned}\sum_k \left\|\mathbf{x}^D_{u_k,u_{k+1}}\right\|^p &\leq 3^{p-1}\sum_k\left\|\mathbf{x}^D_{u_k^D,u_{k+1,D}}\right\|^p + 3^{p-1}\sum_k\left[\left|x^D_{u_k,u_k^D}\right|^p + \left|x^D_{u_{k+1,D},u_k}\right|^p\right] \\ &\leq 3^{p-1}\left(\max_{(s_k)\subset D}\sum_k\left\|\mathbf{x}^D_{s_k,s_{k+1}}\right\|^p\right) + 3^{p-1}\left|x^D\right|_{p\text{-var};[0,T]} \\ &\leq 3^{p-1}\left(\max_{(s_k)\subset D}\sum_k\left\|\mathbf{x}^D_{s_k,s_{k+1}}\right\|^p\right) + C3^{p-1}\left|x\right|_{p\text{-var};[0,T]}.\end{aligned}$$

$\blacksquare$

Theorem 17 *Let F be a moderate function, $\mathbf{M} \in \mathcal{M}_{0,loc}^{c}\left(G^{2}\left(\mathbb{R}^{d}\right)\right)$, and $|\cdot|$, $\|\cdot\|$ continuous homogonous norm on $\mathbb{R}^{d}, G^{2}\left(\mathbb{R}^{d}\right)$ respectively. Then there exists a constant $C = C\left(p, F, d, |\cdot|, \|\cdot\|\right)$ so that for all dissections D of $[0,T]$,*

$$\mathbb{E}\left(F\left(\left\|\mathbf{M}^{D}\right\|_{p-var;[0,T]}\right)\right) \leq C\mathbb{E}\left(F\left(\left|\langle M\rangle_{T}\right|^{1/2}\right)\right).$$

Proof. From Lemma 16, $\left\|\mathbf{M}^{D}\right\|_{p\text{-var};[0,T]}$ is bounded by $C\,|M|_{p\text{-var};[0,T]}$ plus

$$C\left(\max_{(s_k)\subset D}\sum_{k}\left\|\mathbf{M}^{D}_{s_k,s_{k+1}}\right\|^{p}\right)^{1/p} \leq C\left(\max_{(s_k)\subset D}\sum_{k}\left\|\mathbf{M}_{s_k,s_{k+1}}\right\|^{p}\right)^{1/p}$$
$$+C\left(\max_{(s_k)\subset D}\sum_{k} d\left(\mathbf{M}_{s_k,s_{k+1}},\mathbf{M}^{D}_{s_k,s_{k+1}}\right)^{p}\right)^{1/p}.$$

Trivially, $|M|_{p\text{-var};[0,T]} \leq \|\mathbf{M}\|_{p\text{-var};[0,T]}$ and with a new constant C,

$$\left\|\mathbf{M}^{D}\right\|_{p\text{-var};[0,T]} \leq C\,\|\mathbf{M}\|_{p\text{-var};[0,T]}+C\left(\max_{(s_k)\subset D}\sum_{k} d\left(\mathbf{M}_{s_k,s_{k+1}},\mathbf{M}^{D}_{s_k,s_{k+1}}\right)^{p}\right)^{1/p}.$$

For fixed k, there are $i<j$ so that $s_k = t_i$ and $s_{k+1} = t_j$. Then

$$\begin{aligned}
\mathbf{M}_{s_k,s_{k+1}} &= \bigotimes_{l=i}^{j-1}\exp\left(M_{t_l,t_{l+1}} + A_{t_l,t_{l+1}}\right)\\
\mathbf{M}^{D}_{s_k,s_{k+1}} &= \bigotimes_{l=i}^{j-1}\exp\left(M_{t_l,t_{l+1}}\right).
\end{aligned}$$

Hence, $d\left(\mathbf{M}_{s_k,s_{k+1}},\mathbf{M}^{D}_{s_k,s_{k+1}}\right)$ equals

$$\left\|\mathbf{M}^{-1}_{s_k,s_{k+1}}\otimes\mathbf{M}^{D}_{s_k,s_{k+1}}\right\| = \left\|\exp\left(\sum_{l=i}^{j-1}A_{t_l,t_{l+1}}\right)\right\| \leq C\left|\sum_{l=i}^{j-1}A_{t_l,t_{l+1}}\right|^{1/2}. \tag{4}$$

The key idea is to introduce the (vector-valued) discrete-time martingale

$$Y_j = \sum_{l=0}^{j-1}A_{t_l,t_{l+1}} \in so\left(d\right).$$

From (4) and equivalence of homogenous norms we have

$$\max_{(s_k)\subset D}\sum_{k} d\left(\mathbf{M}_{s_k,s_{k+1}},\mathbf{M}^{D}_{s_k,s_{k+1}}\right)^{p} \leq C\max_{\{i_1,\dots,i_n\}\subset\{1,\dots,\#D\}}\sum_{k}\left|Y_{i_{k+1}} - Y_{i_k}\right|^{p/2},$$

which leads to

$$\begin{aligned}\left\|\mathbf{M}^{D}\right\|_{p\text{-var}} &\leq C\left\|\mathbf{M}\right\|_{p\text{-var}}+C\sqrt{\left(\max_{\{i_1,\ldots,i_n\}\subset\{1,\ldots,\#D\}}\sum_{k}\left|Y_{i_{k+1}}-Y_{i_k}\right|^{p/2}\right)^{2/p}}\\ &= C\left\|\mathbf{M}\right\|_{p\text{-var}}+C\sqrt{|Y|_{p/2\text{-var}}}.\end{aligned}$$

Using basic properties of moderate functions we have

$$\begin{aligned}\mathbb{E}\left[F\left(\left\|\mathbf{M}^{D}\right\|_{p\text{-var}}\right)\right] &\leq C\mathbb{E}\left[F\left(\left\|\mathbf{M}\right\|_{p\text{-var}}\right)\right]+C\mathbb{E}\left[F\left(\sqrt{|Y|_{p/2\text{-var}}}\right)\right]\\ &= C\mathbb{E}\left[F\left(\left\|\mathbf{M}\right\|_{p\text{-var}}\right)\right]+C\mathbb{E}\left[F\circ\sqrt{\cdot}\left(|Y|_{p/2\text{-var}}\right)\right].\end{aligned}$$

Note that $F\circ\sqrt{\cdot}$ is moderate since F is moderate. Let $2<p'<p<3$. Then $1<p'/2\leq p/2\leq 2$ and and Lemma 8 yields

$$\begin{aligned}\mathbb{E}\left[F\circ\sqrt{\cdot}\left(|Y|_{p/2\text{-var}}\right)\right] &\leq \mathbb{E}\left[F\circ\sqrt{\cdot}\left(\left(\sum_{l}|Y_{l+1}-Y_{l}|^{p'/2}\right)^{2/p'}\right)\right]\\ &= \mathbb{E}\left[F\circ\sqrt{\cdot}\left(\left(\sum_{l}\left|A_{t_l,t_{l+1}}\right|^{p'/2}\right)^{2/p'}\right)\right]\\ &\leq \mathbb{E}\left[F\left(\left(\sum_{l}\left\|\mathbf{M}_{t_l,t_{l+1}}\right\|^{p'}\right)^{1/p'}\right)\right]\\ &\leq \mathbb{E}\left[F\left(\left\|\mathbf{M}\right\|_{p'\text{-var};[0,T]}\right)\right].\end{aligned}$$

Combing the last two estimates and using Theorem 14 (with $p'=1+p/2>2$ and p respectively) gives

$$\begin{aligned}\mathbb{E}\left[F\left(\left\|\mathbf{M}^{D}\right\|_{p\text{-var};[0,T]}\right)\right] &\leq C\mathbb{E}\left[F\left(\left\|\mathbf{M}\right\|_{p\text{-var};[0,T]}\right)\right]+C\mathbb{E}\left[F\left(\left\|\mathbf{M}\right\|_{p'\text{-var};[0,T]}\right)\right]\\ &\leq 2C\mathbb{E}\left(F\left(|\langle M\rangle_T|^{1/2}\right)\right).\end{aligned}$$

■

Remark 18 *We don't expect a lower BDG bound uniformly over all dissections D of $[0,T]$. For instance,*

$$C^{-1}\mathbb{E}\left(F\left(|\langle M\rangle_T|^{1/2}\right)\right)\leq\mathbb{E}\left(F\left(\left|M^{D}\right|_{\infty;[0,T]}\right)\right)$$

can't hold since $D=\{0,T\}$ implies $M^{D}_{\infty;[0,T]}=|M_T|$ and for $F(x)=x$ we would control

$$\mathbb{E}\left(|M|_{\infty;[0,T]}\right)\sim\mathbb{E}\left(\left|\langle M\rangle_T^{1/2}\right|\right)$$

in terms of $\mathbb{E}(|M_T|)$ which is Doob's L^q maximal inequality with $q=1$. But, as is well known, one needs $q>1$ for Doob's L^q-inequality to hold true.

Let us now bound the supremum distance between $\mathbf{M}$ and $\mathbf{M}^D$:

Lemma 19 *Assume that M is a martingale such that*

$$|M|_{\infty;[0,T]} \in L^q(\Omega) \text{ for some } q \geq 1. \tag{5}$$

If D^n is a sequence of subdivisions whose time steps tends to 0 when n tends to ∞, then $d_{\infty;[0,T]}\left(\mathbf{M}, \mathbf{M}^{D_n}\right)$ converges to 0 in L^q.

Remark 20 *If $q > 1$, Doob's maximal inequality implies that (5) holds for any L^q-martingale.*

Proof of Lemma 19. As in the proof of Theorem 17, equation (4) more specifically, we have that when $t = t_i \in D$

$$d\left(\mathbf{M}_t, \mathbf{M}_t^D\right) \leq C \left|\sum_{k=0}^{i-1} A_{t_k,t_{k+1}}\right|^{1/2}.$$

Next, consider $t \in [t_i, t_{i+1}]$ for some i. The path $M_\cdot^D$ restricted to $[t_i, t_{i+1}]$ is a straight line with no area, hence

$$\mathbf{M}_{t_i,t}^D = \exp\left(\frac{t-s}{t_{i+1}-t_i} M_{t_i,t_{i+1}}\right).$$

and

$$\begin{aligned} d\left(\mathbf{M}_{t,}, \mathbf{M}_t^D\right) &= d\left(\mathbf{M}_{t_i} \otimes \mathbf{M}_{t_i,t}, \mathbf{M}_{t_i}^D \otimes \mathbf{M}_{t_i,t}^D\right) \\ &= \left\|\left(\mathbf{M}_{t_i,t}^D\right)^{-1} \otimes \left(\mathbf{M}_{t_i}^D\right)^{-1} \otimes \mathbf{M}_{t_i} \otimes \mathbf{M}_{t_i,t}\right\| \\ &\leq \left\|\left(\mathbf{M}_{t_i,t}^D\right)\right\| + \left\|\left(\mathbf{M}_{t_i}^D\right)^{-1} \otimes \mathbf{M}_{t_i}\right\| + \left\|\mathbf{M}_{t_i,t}\right\| \\ &\leq 2 \sup_{u,v \in [t_i,t_{i+1}]} \left\|\mathbf{M}_{u,v}\right\| + C \max_{i,j} \left|\sum_{l=i}^{j-1} A_{t_l,t_{l+1}}\right|^{1/2}. \end{aligned}$$

For the L^q convergence, because $\mathbf{M}$ is almost surely continuous (in fact, uniformly continuous on the compact $[0,T]$)

$$\max_{i=0,\ldots,\#D-1} \sup_{s,t \in \left[t_i^n, t_{i+1}^n\right]} \left\|\mathbf{M}_{t_i,t_{i+1}}\right\| \to 0 \text{ a.s.}$$

Hence, by dominated convergence,

$$\lim_{|D_n| \to 0} \mathbb{E}\left(\max_{i=0,\ldots,\#D-1} \sup_{s,t \in \left[t_i^n, t_{i+1}^n\right]} \left\|\mathbf{M}_{t_i,t_{i+1}}\right\|^q\right) = 0.$$

With Y defined as in the proof of Theorem 17,

$$\max_{i,j}\left|\sum_{l=i}^{j-1}A_{t_l,t_{l+1}}\right|^{1/2}\leq C\left[\left(\max_{\{i_1,\dots,i_n\}\subset\{1,\dots,\#D\}}\sum_k\left|Y_{i_{k+1}}-Y_{i_k}\right|^{p/2}\right)^{2/p}\right]^{1/2}$$

the computation given therein with $F(x)=x^q$ shows

$$\begin{aligned}\mathbb{E}\left(\max_{i,j}\left|\sum_{l=i}^{j-1}A_{t_l,t_{l+1}}\right|^{q/2}\right) &\leq C\mathbb{E}\left[F\circ\sqrt{\left(\max_{\{i_1,\dots,i_n\}\subset\{1,\dots,\#D\}}\sum_k\left|Y_{i_{k+1}}-Y_{i_k}\right|^{p/2}\right)^{2/p}}\right]\\ &\leq C\mathbb{E}\left[F\left(\left(\sum_{l:t_l\in D_n}\left\|\mathbf{M}_{t_l,t_{l+1}}\right\|^q\right)^{1/q}\right)\right]\\ &= \mathbb{E}\left[\left(\sum_{l:t_l\in D_n}\left\|\mathbf{M}_{t_l,t_{l+1}}\right\|^q\right)\right].\end{aligned}$$

Bu this last expression tends to zero, combining the bounded convergence theorem with a.s. convergence

$$\lim_{n\to\infty}\sum_{l:t_l\in D_n}\left\|\mathbf{M}_{t_l,t_{l+1}}\right\|^q=0.$$

Indeed, this follows from $\mathbf{M}\in C^{0,q\text{-var}}$ since $q>2$ and using the usual squeezing argument. To show L^q convergence with respect to $d_\infty=d_{\infty;[0,T]}$, we also write $\|\cdot\|_\infty=\|\cdot\|_{\infty;[0,T]}$ here, recall that

$$d_\infty\left(\mathbf{M},\mathbf{M}^D\right)\leq\sup_{t\in[0,T]}d\left(\mathbf{M}_t,\mathbf{M}_t^D\right)+c\left|\|\mathbf{M}\|_\infty\sup_{t\in[0,T]}d\left(\mathbf{M}_t,\mathbf{M}_t^D\right)\right|^{1/2}.$$

We just showed that $\sup_{t\in[0,T]}d\left(\mathbf{M}_t,\mathbf{M}_t^D\right)\to 0$ in L^q. Then

$$\begin{aligned}&\mathbb{E}\left(\left|\|\mathbf{M}\|_\infty\sup_{t\in[0,T]}d\left(\mathbf{M}_t,\mathbf{M}_t^D\right)\right|^{q/2}\right)\\ \leq\ &\left(\mathbb{E}\left(\left|\|\mathbf{M}\|_\infty\right|^q\right)\right)^{1/2}\left(\mathbb{E}\left(\left|\sup_{t\in[0,T]}d\left(\mathbf{M}_t,\mathbf{M}_t^D\right)\right|^q\right)\right)^{1/2}.\end{aligned}$$

(Note that by the our BDG inqualities

$$\mathbb{E}\left(\left|\|\mathbf{M}\|_{\infty;[0,T]}\right|^q\right)\leq C\mathbb{E}\left(\left|\langle M\rangle_T\right|^q\right)\leq C\mathbb{E}\left(\left||M|_{\infty;[0,T]}\right|^q\right)$$

and the last expression is finite by assumption.) ■

Theorem 21 *Let M be as in Lemma 19. Then, $d_{p\text{-}var;[0,T]}\left(\mathbf{M}^D,\mathbf{M}\right)$ converges to 0 in L^q. If M is a local martingale, then convergence holds in probability.*

Proof. The result for the local martingale will hold if the first result holds, by a localisation argument that we leave to the reader. We already saw that L^q-convergence holds w.r.t. $d_\infty = d_{\infty;[0,T]}$. To go further, writing $d_{p\text{-var}} \equiv d_{p\text{-var};[0,T]}$, we use the interpolation formula

$$d_{p\text{-var}}\left(\mathbf{M},\mathbf{M}^D\right) \le C d_\infty\left(\mathbf{M},\mathbf{M}^D\right)^{1-\frac{p'}{p}}\left(\|\mathbf{M}\|_{p'-var}^{\frac{p'}{p}} + \left\|\mathbf{M}^D\right\|_{p'-var}^{\frac{p'}{p}}\right),\ \ 2<p'<p.$$

Hence,

$$\mathbb{E}\left(\left|d_{p-var}\left(\mathbf{M}^D,\mathbf{M}\right)\right|^q\right) \le C\mathbb{E}\left(\left(\|\mathbf{M}\|_{p'-var}^{q\frac{p'}{p}} + \left\|\mathbf{M}^D\right\|_{p'-var}^{q\frac{p'}{p}}\right) d_\infty\left(\mathbf{M},\mathbf{M}^D\right)^{q\left(1-\frac{p'}{p}\right)}\right)$$

Using Hölder with conjugate exponents $1/\left(p'/p\right)$ and $1/\left(1-p'/p\right)$ gives

$$\mathbb{E}\left(\left|d_{p-var}\left(\mathbf{M}^D,\mathbf{M}\right)\right|^q\right) \le C\mathbb{E}\left(\|\mathbf{M}\|_{p'-var}^{q} + \left\|\mathbf{M}^D\right\|_{p'-var}^{q}\right)^{p'/p}\left[\mathbb{E}\left(d_\infty\left(\mathbf{M},\mathbf{M}^D\right)^q\right)\right]^{1-p'/p}.$$

But now it suffices to remark, using our BDG estimates, that

$$\mathbb{E}\left(\|\mathbf{M}\|_{p'-var;[0,T]}^{q}\right),\mathbb{E}\left(\left\|\mathbf{M}^D\right\|_{p'-var;[0,T]}^{q}\right) \le C\mathbb{E}\left(\left|\langle M\rangle_T\right|^{q/2}\right) \le C\mathbb{E}\left(\left||M|_{\infty;[0,T]}\right|^q\right)$$

and the last term is finite by assumption. ■

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